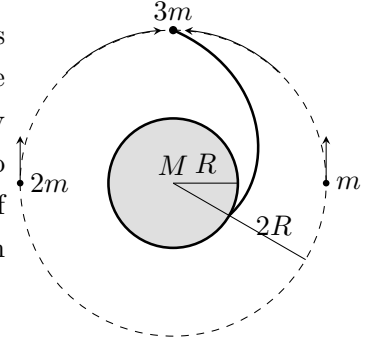


1. Two satellites of mass  $m$ , and  $2m$ , are on a circular orbit of radius  $2R$  around a planet of mass  $M$  and radius  $R$ . The satellites are moving on opposite senses on the circular orbit and collide completely inelastically forming a body of mass  $3m$ . The resulting body falls to the planet's surface following an elliptical trajectory. The masses of the satellites are small enough for the gravitational attraction between them to be negligible.



[8 Pts.]

- (a) Find the speed of the satellites before the collision.

For a circular orbit of radius  $r = 2R$ , the gravitational force provides the centripetal force

$$\frac{GMm}{(2R)^2} = \frac{mv^2}{2R} \Rightarrow v = \sqrt{\frac{GM}{2R}}.$$

[8 Pts.]

- (b) What is the mechanical energy lost in the collision in terms of  $m$ ,  $M$ ,  $G$ , and  $R$ ?

Taking the direction of motion of the satellite of mass  $2m$  as positive, the total momentum before the collision is

$$p_i = (2m)v - mv = mv.$$

If its speed immediately after collision is  $V$ , then by conservation of momentum,

$$3mV = mv \quad \text{Therefore,} \quad V = \frac{v}{3}.$$

Before collision:  $K_i = \frac{1}{2}mv^2 + \frac{1}{2}(2m)v^2 = \frac{3}{2}mv^2$ . After collision:  $K_f = \frac{1}{2}(3m)V^2 = \frac{1}{6}mv^2$ . Hence,

$$\Delta E = K_i - K_f = \left(\frac{3}{2} - \frac{1}{6}\right)mv^2 = \frac{2}{3}\frac{GMm}{R}.$$

[9 Pts.]

- (c) What is the speed of the mass as it collides with the planet?

The total energy immediately after collision is

$$E_i = \frac{1}{2}(3m)V^2 - \frac{GM(3m)}{2R} = -\frac{17GMm}{12R}.$$

Let the speed at impact with the planet be  $u$ . At the surface, the total energy is

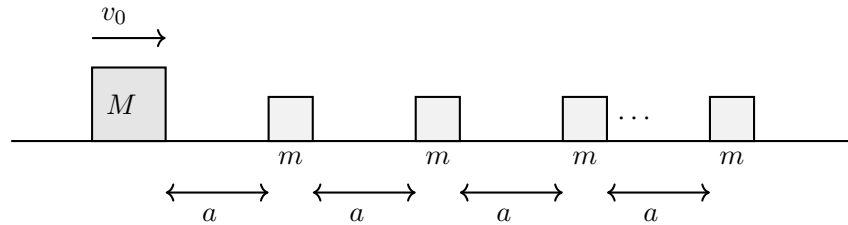
$$E_f = \frac{1}{2}(3m)u^2 - \frac{GM(3m)}{R} = \frac{3}{2}mu^2 - \frac{3GMm}{R}.$$

By conservation of mechanical energy,

$$-\frac{17GMm}{12R} = \frac{3}{2}mu^2 - \frac{3GMm}{R} \Rightarrow u = \sqrt{\frac{19GM}{18R}}.$$

2. A block of mass  $M$  moves without friction along a horizontal surface with initial speed  $v_0$ . In front of it are  $n$  small masses, each of mass  $m$ , initially at rest on the same straight line. The small masses are arranged so that the distance between successive masses is  $a$ , as shown in the figure. All masses can be regarded as point particles.

The block collides in sequence with each small mass. Each collision is perfectly inelastic, and the small mass sticks to the block.



[5 Pts.]

- (a) Find the speed of the combined block immediately after the first collision.

Momentum is conserved. If  $v_1$  is the speed immediately after the first collision, then

$$Mv_0 = (M + m)v. \quad \text{Hence,} \quad v = \frac{M}{M + m}v_0.$$

[5 Pts.]

- (b) Determine the speed of the system immediately after the  $n$ th collision.

After the  $n$ th collision, total mass is  $M + nm$ . Conservation of momentum gives

$$Mv_0 = (M + nm)v_n, \quad \text{Therefore,} \quad v_n = \frac{M}{M + nm}v_0.$$

[5 Pts.]

- (c) Calculate the total kinetic energy lost after the  $n$ th collision.

The initial kinetic energy is  $K_i = \frac{1}{2}Mv_0^2$ . After the  $n$ th collision, the final kinetic energy is

$$K_f = \frac{1}{2}(M + nm)v_n^2 = \frac{1}{2} \frac{M^2}{M + nm}v_0^2.$$

Hence the total kinetic energy lost is

$$\Delta E = K_i - K_f = \frac{1}{2}Mv_0^2 - \frac{1}{2} \frac{M^2}{M + nm}v_0^2 = \frac{1}{2} \frac{Mnm}{M + nm}v_0^2.$$

[10 Pts.]

- (d) Find the time taken between the first collision and the  $n$ th collision.

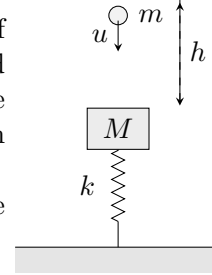
Between the first and second collisions, the moving system travels a distance  $a$  with constant speed  $v_1$ , so the time taken is  $t_1 = \frac{a}{v_1}$ . Between the second and third collisions, the speed is  $v_2$ , so the time taken is  $t_2 = \frac{a}{v_2}$ . Therefore, the total time between the first collision and the  $n$ th collision is

$$t = \sum_{l=1}^{n-1} \frac{a}{v_l} = \sum_{r=1}^{n-1} \frac{a(M + lm)}{Mv_0} = \frac{a}{Mv_0} \sum_{l=1}^{n-1} (M + lm).$$

Since  $\sum_{l=1}^{n-1} l = \frac{n(n-1)}{2}$ ,

$$t = \frac{a(n-1)}{v_0} \left[ 1 + \frac{mn}{2M} \right].$$

3. A block of mass  $M$  rests in equilibrium on a light vertical spring of spring constant  $k$ , as shown in the figure below. The spring is fixed to the ground. A small ball of mass  $m$  is at a height  $h$  above the block and is moving downward with speed  $u$ . The ball collides with the block and sticks to it.



Assume that the collision is perfectly inelastic and that air resistance is negligible.

- [5 Pts.] (a) Find the speed of the ball immediately before the collision.

Mechanical energy is conserved:

$$\frac{1}{2}mu^2 + mgh = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{u^2 + 2gh}.$$

- [5 Pts.] (b) Write down an expression for the speed of the combined ball-block system immediately after the collision in terms of  $m$ ,  $M$ ,  $u$ ,  $h$ ,  $g$ .

Let  $V$  be the speed of the combined ball-block system immediately after the collision. Momentum is conserved:

$$mv = (M + m)V \Rightarrow V = \frac{m}{M + m}v = \frac{m}{M + m}\sqrt{u^2 + 2gh}.$$

- [7 Pts.] (c) Calculate the net work done on the ball-block system from immediately after the collision until the instant of maximum compression, and explain the sign of your answer.

$$W_{\text{net}} = \Delta K = 0 - \frac{1}{2}(M + m)V^2 = -\frac{m^2}{2(M + m)}(u^2 + 2gh).$$

The work is negative because the kinetic energy of the ball-block system decreases from a positive value immediately after the collision to zero at maximum compression.

- [8 Pts.] (d) Let  $x_{\text{max}}$  be the additional compression of the spring measured from the initial equilibrium position of the block. Derive an equation relating  $x_{\text{max}}$  to  $m$ ,  $M$ ,  $u$ ,  $h$ ,  $g$ , and  $k$ . (You do not need to solve it.)

Let  $x_{\text{max}}$  be the additional compression measured from the initial equilibrium position of the block.

- the work done by gravity is

$$W_g = (M + m)g x_{\text{max}},$$

- the work done by the spring is

$$W_s = -Mg x_{\text{max}} - \frac{1}{2}kx_{\text{max}}^2.$$

Therefore, the net work is

$$W_{\text{net}} = (M + m)g x_{\text{max}} - Mg x_{\text{max}} - \frac{1}{2}kx_{\text{max}}^2 = mg x_{\text{max}} - \frac{1}{2}kx_{\text{max}}^2.$$

Using ,  $W_{\text{net}} = \Delta K = -\frac{1}{2}(M + m)V^2$ . Then,

$$\frac{1}{2}kx_{\text{max}}^2 - mg x_{\text{max}} - \frac{m^2}{2(M + m)}(u^2 + 2gh) = 0.$$

4. Two blocks are on a frictionless horizontal tabletop. Block  $A$  has mass  $m$  and slides at a speed  $v$  toward block  $B$  of mass  $4m$ , which is initially at rest. A coil spring, which obeys Hooke's law and has spring constant  $k$ , is attached to block  $B$  in such a way that it will be compressed when struck by block  $A$ , as shown in the figure. (Ignore the mass of the spring.)



[8 Pts.]

- (a) What will be the velocities of the two blocks at the instant of maximum compression?  
 At the instant of maximum compression, the two blocks have the same velocity, let this common velocity be  $v'$ .  
 By conservation of linear momentum,

$$mv = (m + 4m)v' \Rightarrow v' = \frac{v}{5} \text{ to the right.}$$

[10 Pts.]

- (b) What will be the maximum compression of the spring?  
 Use conservation of mechanical energy  $E_i = E_f$ .

$$\frac{1}{2}kx_{\max}^2 + \frac{1}{2}5mv'^2 = \frac{1}{2}mv^2.$$

So

$$\frac{1}{2}kx_{\max}^2 = \left(\frac{1}{2} - \frac{1}{10}\right)mv^2 = \frac{2}{5}mv^2.$$

Hence,

$$kx_{\max}^2 = \frac{4}{5}mv^2, \Rightarrow x_{\max} = 2v\sqrt{\frac{m}{5k}}$$

[7 Pts.]

- (c) It is observed that long after the collision, when the two blocks are separated, the velocity of block  $A$  is  $v'_A = -3v/5$ . What is the magnitude and the direction of the impulse delivered to block  $A$  by block  $B$ ?  
 The impulse delivered to block  $A$  by block  $B$  equals the change in momentum of block  $A$ .

$$J = \Delta p = p_f - p_i = m\left(-\frac{3v}{5}\right) - mv = -\frac{8mv}{5}.$$

to the left (opposite the initial motion of block  $A$ ).